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THREE-TERM MACHIN-TYPE FORMULAE

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ABSTRACT. We shall show that there exist only finitely many nondegenerate three-term Machin-type formulae and give explicit upper bounds for the sizes of variables.

1. INTRODUCTION

The Machin's formula

$$(1) \quad 4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \frac{\pi}{4},$$

is well known and have been used to calculate approximate values of π . Analogous formulae $\arctan(1/2) + \arctan(1/3) = \pi/4$, $2 \arctan(1/2) - \arctan(1/7) = \pi/4$ and $2 \arctan(1/3) - \arctan(1/7) = \pi/4$, which are also well known, were attributed to Euler, Hutton and Hermann, respectively. But according to Tweddle [11], these formulae also seem to have been found by Machin.

Several three-term formulae such as $8 \arctan(1/10) - \arctan(1/239) - 4 \arctan(1/515) = \pi/4$ due to Simson in 1723 (see [11]) and $12 \arctan(1/18) + 8 \arctan(1/57) - 5 \arctan(1/239) = \frac{\pi}{4}$ due to Gauss in 1863 also have been known.

More generally, an n -terms Machin-type formula is defined to be an identity of the form

$$(2) \quad y_1 \arctan \frac{1}{x_1} + y_2 \arctan \frac{1}{x_2} + \cdots + y_n \arctan \frac{1}{x_n} = \frac{r\pi}{4}$$

with integers $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n$ and $r \neq 0$.

Theoretical studies of Machin-type formulae have begun with a series of works of Størmer', who proved that the four formulae mentioned above are all two-term ones in 1895 [8] and gave a necessary and sufficient condition for given integers $x_1, x_2, \dots, x_n > 1$ to have a Machin-type formula (2) and 102 three-term ones in 1896 [9]. Størmer asked for other three-term Machin-type formulae and questioned whether there exist infinitely many ones or not. Up to now the only known other nontrivial (i.e. not derived from the three formulae given above) three-term formulae are $5 \arctan(1/2) + 2 \arctan(1/53) + \arctan(1/4443) = 3\pi/4$, $5 \arctan(1/3) - 2 \arctan(1/53) - \arctan(1/4443) = \pi/2$ and $5 \arctan(1/7) + 4 \arctan(1/53) + 2 \arctan(1/4443) = \pi/4$. [12] attributes these formulae to Wrench

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[14] although these formulae cannot be found there. We note that the second and the third formulae follow from the first formula using $\arctan(1/2) + \arctan(1/3) = \pi/4$ and $2\arctan(1/2) - \arctan(1/7) = \pi/4$ respectively.

The purpose of this paper is to answer to Størmer's other question in negative. We shall show that there exist only finitely many three-term Machin-type formulae which does not arise from a linear combinations of three two-term formulae.

Størmer's criterion is essentially as follows: For given integers $x_1, x_2, \dots, x_n > 1$, (2) holds for some integers y_1, y_2, \dots, y_n and $r \neq 0$ if and only if there exist integers $s_{i,j} (i = 1, 2, \dots, n, j = 1, 2, \dots, n-1)$ and Gaussian integers $\eta_1, \eta_2, \dots, \eta_{n-1}$ such that

$$(3) \quad \left[\frac{x_i + \sqrt{-1}}{x_i - \sqrt{-1}} \right] = \left[\frac{\eta_1}{\bar{\eta}_1} \right]^{\pm s_{i,1}} \left[\frac{\eta_2}{\bar{\eta}_2} \right]^{\pm s_{i,2}} \cdots \left[\frac{\eta_{n-1}}{\bar{\eta}_{n-1}} \right]^{\pm s_{i,n-1}}.$$

for $i = 1, 2, \dots, n$.

Writing $m_j = \eta_j \bar{\eta}_j$ for $j = 1, 2, \dots, n-1$, this condition can be reformulated as follows: there exist nonnegative integers $s_{i,j} (i = 1, 2, \dots, n, j = 1, 2, \dots, n)$ with $0 \leq s_{i,n} \leq 1$ such that the equation

$$(4) \quad x_i^2 + 1 = 2^{s_{i,n}} m_1^{s_{i,1}} m_2^{s_{i,2}} \cdots m_{n-1}^{s_{i,n-1}}$$

holds for $i = 1, 2, \dots, n$ and, additionally, $x_i \equiv \pm x_j \pmod{m_k}$ for three indices i, j, k with $x_i^2 + 1 \equiv x_j^2 + 1 \equiv 0 \pmod{m_k}$.

Thus, for given three integers $x_1, x_2, x_3 > 1$, there exist nonzero integers y_1, y_2, \dots, y_n and r such that a three-term Machin-type formula

$$(5) \quad y_1 \arctan \frac{1}{x_1} + y_2 \arctan \frac{1}{x_2} + y_3 \arctan \frac{1}{x_3} = \frac{r\pi}{4}$$

holds if and only if there exist integers $k_i, l_i (i = 1, 2, 3)$ and Gaussian integers η_1, η_2 such that

$$(6) \quad \left[\frac{x_i + \sqrt{-1}}{x_i - \sqrt{-1}} \right] = \left[\frac{\eta_1}{\bar{\eta}_1} \right]^{\pm k_i} \left[\frac{\eta_2}{\bar{\eta}_2} \right]^{\pm l_i}$$

holds for $i = 1, 2, 3$ or, equivalently, writing $m_j = \eta_j \bar{\eta}_j$ for $j = 1, 2, \dots, n-1$ and choosing $v_i \in \{0, 1\}$ appropriately, the equation

$$(7) \quad x_i^2 + 1 = 2^{v_i} m_1^{k_i} m_2^{l_i}$$

holds for $i = 1, 2, 3$ and, additionally, $x_i \equiv \pm x_{i'} \pmod{m_j}$ for two indices i, i' with $x_i^2 + 1 \equiv x_{i'}^2 + 1 \equiv 0 \pmod{m_j}$. Furthermore, (6) implies (5) with $y_1 = \pm k_2 l_3 \pm k_3 l_2, y_2 = \pm k_3 l_1 \pm k_1 l_3$ and $y_3 = \pm k_1 l_2 \pm k_2 l_1$ with appropriate choices of signs.

Now we shall state our result in more detail.

Theorem 1.1. *Assume that $x_1, x_2, x_3, y_1, y_2, y_3$ and r are nonzero integers with $x_1, x_2, x_3 > 1$ and $\{x_1, x_2, x_3\} \neq \{2, 3, 7\}$ satisfying (5) and $m_1, m_2, s_i, k_i, l_i (i = 1, 2, 3)$ are corresponding integers with $m_2 > m_1 > 0$ satisfying (7).*

- I. If $x_i^2 + 1 \geq m_2$ for $i = 1, 2, 3$, then $m_1 < m_2 < 5.19 \cdot 10^{39}$, $x_i < \exp(9.726 \cdot 10^{11})$ and $|y_i| < 2KL < 5.656 \cdot 10^{19}$.
- II. If $x_i^2 + 1 < m_2$ for some i , then $m_1 < 4.14 \cdot 10^{81}$, $m_2 < \exp(9964497.86) < 2.7261 \times 10^{4327526}$, $x_i < \exp(3.18 \cdot 10^{20})$ and $y_i < 2.7 \cdot 10^{31}$.

We use a lower bound for linear forms in three logarithms in order to obtain upper bounds for exponents k_i 's and l_i 's in terms of m_1, m_2 .

These upper bounds themselves do not give finiteness of m_1 and m_2 . However, noting that $r \neq 0$, which gives $|\sum_i y_i \arctan(1/x_i)| \geq \pi/4$, the first case can be easily settled using these upper bounds. In order to settle the second case, we additionally need an upper bound m_2 in terms of m_1 . This can be done using a lower bound for a quantity of the form $y \arctan(1/x) - r\pi/2$, which gives a linear form of two logarithms.

(4) can be seen as a special case of the generalized Ramanujan-Nagell equation

$$(8) \quad x^2 + Ax + B = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n},$$

where A and B are given integers with $A^2 - 4B \neq 0$ and p_1, p_2, \dots, p_n are given primes. Evertse [2] proved that (8) has at most $3 \cdot 7^{4n+6}$ solutions. In the case $n = 2$, the author [15] reduced Evertse's bound $3 \cdot 7^{14}$ to 63.

On the other hand, our result does not give an upper bound for numbers of solutions

$$(9) \quad x^2 + 1 = 2^s p_1^k p_2^l$$

since the case $r = 0$ is not considered. Indeed, Størmer [9] implicitly pointed out that, if $x^2 + 1 = ay$, then

$$(10) \quad \arctan \frac{1}{az - x} - \arctan \frac{1}{az + a - x} = \arctan \frac{1}{az(z+1) - (2z+1)x + y}.$$

Størmer [10] showed that (9) has at most one solution with each fixed combination of parities of s_i, k_i, l_i with zero and nonzero-even distinguished. Although there exist 18 combinations $(0 \mid 1, 0 \mid 1 \mid 2, 0 \mid 1 \mid 2)$, all-even combinations can clearly be excluded and therefore (9) has at most 14 solutions totally.

2. PRELIMINARIES

In this section, we introduce some notation and some basic facts.

For integers N composed of prime factors $\equiv 1 \pmod{4}$, we define $\widehat{\log} N = \log N$ if $N \geq 13$ and $\widehat{\log} 5 = 4 \arctan(1/2)$. If we decompose $N = \eta \bar{\eta}$ in Gaussian integers, then $\log(\eta/\bar{\eta}) \leq (\widehat{\log} N)/2$. We write $\gamma(N) = \widehat{\log} N / \log N$. $\gamma(5) = 1.1523 \cdots$ and $\gamma(N) = 1$ for $N \geq 13$.

Moreover, we define $\widetilde{\log} N$ by $\widetilde{\log} N = \max\{\log N, (1/2.648) + \max 4 \arg(\eta/\bar{\eta}) / \log N\}$, where the inner maximum is taken over all decompositions $N = \eta \bar{\eta}$ with $|\arg \eta| <$

$\pi/4$. We write $\delta(N) = \widetilde{\log} N / \log N$. We see that $\delta(N) = 1$ when $N > 22685$ and there exist exactly 401 integers N such that $\delta(N) > 1$.

For any gaussian integer η , we have an associate η' of η such that $-\pi/4 < \arg \eta' < \pi/4$ and therefore $-\pi/2 < \arg \eta' / \bar{\eta}' < \pi/2$.

We call a formula (2) to be degenerate if

$$(11) \quad \sum_{i \in S} y'_i \arctan \frac{1}{x_i} = \frac{r'\pi}{4}$$

for some proper subset S of $\{1, 2, \dots, n\}$ and integers $y'_i (i \in S)$ and r' which may be zero but not all zero.

From Størmer's result in [8] on two-term Machin-type formulae, the degenerate case only occurs in $\{x_1, x_2, x_3\} = \{2, 3, 7\}$.

3. A LOWER BOUND FOR LINEAR FORMS OF THREE LOGARITHMS

Our argument depends on a lower bound for linear forms of three logarithms. Results in Mignotte's *a kit on linear forms in three logarithms*[6] are rather technical but still worthful to use for the purpose of improving our upper bounds. Proposition 5.2 of [6] applied to the Gaussian rationals gives the following result.

Lemma 3.1. *Let α_1, α_2 and α_3 be three Gaussian rationals $\neq 1$ with absolute value one and assume that the three numbers $\alpha_1, \alpha_2, \alpha_3$ are multiplicatively independent or two of these numbers are multiplicatively independent and the third one is a root of unity, i.e. -1 or $\pm\sqrt{-1}$. Let b_1, b_2 and b_3 be three coprime positive rational integers and*

$$(12) \quad \Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1 - b_3 \log \alpha_3,$$

where the logarithm of each α_i can be arbitrarily determined as long as

$$(13) \quad b_2 |\log \alpha_2| = b_1 |\log \alpha_1| + b_3 |\log \alpha_3| \pm |\Lambda|.$$

We put $d_1 = \gcd(b_1, b_2), d_2 = \gcd(b_2, b_3), b_2 = d_1 b'_2 = d_3 b''_2$. Let $w_i = |\log \alpha_i| = |\arg \alpha_i|$ for each $i = 1, 2, 3$, a_1, a_2 and a_3 be real numbers such that $a_i \geq \max\{4, 5.296w_i + 2h(\alpha_i)\}$ for each $i = 1, 2, 3$ and $\Omega = a_1 a_2 a_3 \geq 100$. Furthermore, put

$$(14) \quad b' = \left(\frac{b'_1}{a_2} + \frac{b'_2}{a_1} \right) \left(\frac{b''_3}{a_2} + \frac{b''_2}{a_3} \right)$$

and $\log B = \max\{0.882 + \log b', 10\}$.

Then, either one of the following holds.

A. *The estimate*

$$(15) \quad \log |\Lambda| > -790.95 \Omega \log^2 B$$

holds.

B. *There exist two nonzero rational integers r_0 and s_0 such that $r_0 b_2 = s_0 b_1$ with $|r_0| \leq 5.61 a_2 \log^{1/3} B$ and $|s_0| \leq 5.61 a_1 \log^{1/3} B$.*

C. *There exist four rational integers r_1, s_1, t_1 and t_2 with $r_1 s_1 \neq 0$ such that*
(16) $(t_1 b_1 + r_1 b_3) s_1 = r_1 b_2 t_2, \gcd(r_1, t_1) = \gcd(s_1, t_2) = 1$

and

(17) $|r_1 s_1| \leq 5.61 \delta a_3 \log^{1/3} B, |s_1 t_1| \leq 5.61 \delta a_1 \log^{1/3} B, |r_1 t_2| \leq 5.61 \delta a_2 \log^{1/3} B,$

where $\delta = \gcd(r_1, s_1)$. Moreover, when $t_1 = 0$ we can take $r_1 = 1$ and then $t_2 = 0$ we can take $s_1 = 1$.

This result is nonsymmetric for three logarithms and, in order to make each b_i positive, we should arrange the order of logarithms. Thus, the application of this result requires a fair amount of computations with many branches of cases.

For convenience, we write h_i for $h(\alpha_i)$. For our purpose, we apply Lemma 3.1 to linear forms of two logarithms and $\pi\sqrt{i}/2 = \log \sqrt{-1}$. In this special case, we may assume that (i) $\log \alpha_2 = \pi/2$ or (ii) $\log \alpha_3 = \pi/2$ by exchanging (α_1, b_1) and (α_3, b_3) . Thus, there exist six cases: A. i, A. ii, B. i, B. ii, C. i, C. ii.

In Case A, (15) gives a desired lower bounds. In cases B and C, we can reduce Λ into a linear form of two logarithms and apply results of [3]. Here, we shall discuss only in the case C. i.

We put $r_1 = \delta r_0, s_1 = \delta s_0$, which immediately yields that $\gcd(r_0, s_0) = 1$. Dividing (16) by δ , we have

$$(18) \quad s_0 t_1 b_1 + r_0 t_2 b_2 + \delta r_0 s_0 b_3 = 0.$$

From this, we see that r_0 divides b_1 and s_0 divides b_2 . Put $b_1 = r_0 u_1, b_2 = s_0 u_2$. Dividing (18) by $r_0 s_0$, we have

$$(19) \quad t_1 u_1 + t_2 u_2 + \delta b_3 = 0.$$

Now we obtain

$$(20) \quad \delta \Lambda = u_2 \log \alpha_5 - u_1 \log \alpha_6,$$

where $\alpha_5 = \alpha_2^{s_1} \alpha_3^{t_2}, \alpha_6 = \alpha_1^{r_1} \alpha_3^{-t_1}$. Moreover,

$$(21) \quad |s_0 t_1| \leq 5.61 a_1 \log^{1/3} B, |r_0 t_2| \leq 5.61 a_2 \log^{1/3} B, |\delta r_0 s_0| \leq 5.61 a_3 \log^{1/3} B.$$

Taking

$$a_5 = \max \left\{ |t_2| h_3, |t_2| w_3 + \frac{|s_1| \pi}{2} \right\}, a_6 = \max \{ |r_1| h_1 + |t_1| h_3, |r_1| w_1 + |t_1| w_3 \}$$

and

$$b'' = \frac{|u_1|}{a_5} + \frac{|u_2|}{a_6} \leq \frac{b_1}{|s_0| a_5} + \frac{b_2}{|s_0| a_6},$$

Corollaire 1 of [3] gives

$$(22) \quad \log |\delta \Lambda| \geq -30.9 \max \{ \log^2 b'', 441 \} a_5 a_6.$$

TABLE 1. Constants in (24)

Case	C	μ_1	μ_2	μ	ν_1	ν_2	β	τ
A, i	$28962f_1(m_1, m_2)$	1	1	2	1/2	1/2	$\frac{1}{\log m_1} + \frac{1}{5.296\pi}$	2.351
A, ii	$28962f_1(m_1, m_2)$	1	1	2	0	1/2	$\sqrt{\frac{1}{2.648\pi} + \frac{1}{\log m_1}}$	2.393
B, i	$460.63f_3(m_1, m_2)$	1	1	7/3	1/2	1/2	$\frac{1}{\log m_1} + \frac{1}{5.296\pi}$	4.574
					0	0	2	3.967
B, ii	$127.408f_4(m_1, m_2)$	1	1	7/3	0	1/2	$\sqrt{\frac{1}{2.648\pi} + \frac{1}{\log m_1}}$	4.902
					0	1	126.844	2.838
C, i	$6631g_5(m_1, m_2)$	1	2	8/3	1/2	3/2	$\frac{1}{\log m_1} + \frac{1}{5.296\pi}$	4.529
					0	1	$\pi/2 + 2$	4.025
C, ii	$27574\delta(m_1)\delta(m_2)$	1	1	8/3	0	1/2	$\sqrt{\frac{1}{2.648\pi} + \frac{1}{\log m_1}}$	4.475
					0	0	2	4.006

4. UPPER BOUNDS FOR EXPONENTS

In this section, we shall prove upper bounds for exponents in (6) or, equivalently, (7).

Lemma 4.1. *Let η_1 and η_2 be Gaussian integers with $-\pi/2 < \arg \eta_i/\bar{\eta}_i < \pi/2$ and $m_i = \eta_i\bar{\eta}_i > 1$ for $i = 1, 2$ with $m_2 > m_1$ both odd.*

We set

$$\begin{aligned}
 f_1(m_1, m_2) &= \left(1 + \frac{5.296\pi}{\log m_1}\right) \left(1 + \frac{5.296\pi}{\log m_2}\right), \\
 f_3(m_1, m_2) &= \max\{\delta(m_1), \gamma(m_1)\delta(m_2)\}, \\
 f_4(m_1, m_2) &= \frac{1}{2} \left(\left(1 + \frac{5.296\theta_1}{\log m_1}\right) \left(1 + \frac{10.98\theta_2}{\log m_2}\right) + \left(1 + \frac{10.98\theta_1}{\log m_1}\right) \left(1 + \frac{5.296\theta_2}{\log m_2}\right) \right), \\
 f_5(m_1, m_2) &= 1 + \frac{2.648\pi(\widehat{\log m_1 + \log m_2})}{\widehat{\log m_1 \log m_2}}
 \end{aligned}$$

and

$$g_5(m_1, m_2) = f_5(m_1, m_2)\gamma(m_1)\delta(m_2).$$

If x, e_1, e_2 are nonnegative integers such that

$$(23) \quad \left[\frac{x + \sqrt{-1}}{x - \sqrt{-1}} \right] = \left[\frac{\eta_1}{\bar{\eta}_1} \right]^{\pm e_1} \left[\frac{\eta_2}{\bar{\eta}_2} \right]^{\pm e_2},$$

then we have

$$(24) \quad e_1 \log m_1 + e_2 \log m_2 < 2\tau C \log^{\mu_1} m_1 \log^{\mu_2} m_2 \log^{\mu} Y$$

with $(C, \mu_1, \mu_2, \mu, \nu_1, \nu_2, \beta, \tau)$ taken from one of ten rows in Table 1 and $Y = 2C\beta \log^{\nu_1} m_1 \log^{\nu_2} m_2$.

Proof. From the result of [4], $x^2 + 1 = m^t$ with $x > 0, t > 1$ has no solution. Théorème 8 of [10] shows that $x^2 + 1 = 2m^t$, then t must be a power of two. By Ljunggren's result [5], the only integer solution of $x^2 + 1 = 2m^4$ with $x, m > 1$ is $(x, m) = (239, 13)$ (Easier proofs of Ljunggren's result have been obtained by Steiner and Tzanakis [7] and Wolfskill [13]). Thus, we may assume that $e_1 e_2 \neq 0$ since $e_i = 0$ implies that $e_{3-i} = 1, 2$ or 4 . Furthermore, we may assume that $m_1^{e_1} m_2^{e_2} > 10^{20}$.

We can decompose $m_i = \eta_i \bar{\eta}_i$ in a way such that $-\pi/4 < \arg \eta' < \pi/4$. We put $\xi_i = \eta_i / \bar{\eta}_i$ and write $\theta_i = |\arg \xi_i| = |\log \xi_i|$, so that $\theta_i < \pi/2$.

Now $\Lambda = \log[(x + \sqrt{-1})/(x - \sqrt{-1})]$ can be represented as a linear form of three logarithms

$$(25) \quad \Lambda = \pm e_1 \log \xi_1 \pm e_2 \log \xi_2 \pm \frac{e_3 \pi \sqrt{-1}}{2}$$

for an appropriate integer $e_3 \geq 0$. Moreover, we can easily see that

$$(26) \quad \log |\Lambda| < -\log x < -\frac{e_1 \log m_1 + e_2 \log m_2}{2} + 10^{-9}.$$

Applying Lemma 3.1 and some technical argument in each of six cases, which are too complicated to describe here, we are led to 24. This proves the lemma. \square

5. PROOF OF THE THEOREM

Let $x_1, x_2, x_3, y_1, y_2, y_3$ and r be integers with $x_1, x_2, x_3 > 1, r \neq 0$ satisfying (5) and $m_1, m_2, s_i, k_i, l_i (i = 1, 2, 3)$ be corresponding integers, η_1, η_2 be gaussian integers satisfying (6) and (7). We write $K = \max k_i$ and $L = \max l_i$. We may assume that $\{x_1, x_2, x_3\} \neq \{2, 3, 7\}$. From a note in the preliminaries, this implies that (5) is nondegenerate.

We have two cases: I. $x_1^2 + 1 \geq m_2$ and II. $x_1^2 + 1 < m_2$.

Case I. In this case, $x_i \geq \sqrt{m_2 - 1}$ for $i = 1, 2, 3$. Since (5) is nondegenerate, $u_i = 0$ for at most one index i . In the case there exists such an index i , we may assume that $i = 1$. Since $x_2^2 + 1 \equiv x_3^2 + 1 \equiv 0 \pmod{m_2}$, Størmer's criterion implies that $x_2 > m_2/2$ or $x_3 > m_2/2$.

It immediately follows from (5) with $r \neq 0$ that

$$(27) \quad \frac{|y_1| + |y_2|}{\sqrt{m_2 - 1}} + \frac{2|y_3|}{m_2} > \frac{\pi}{4}.$$

Since $|y_1| \leq k_2 l_3 + k_3 l_2 \leq 2TU$ and so on, we have $m_2 < (4(2 + 10^{-8})KL/\pi)^2 < 6.49(KL)^2$.

Combining with Lemma 4.1, we have $m_1 < m_2 < 5.19 \cdot 10^{39}$, $|y_i| < 2KL < 5.656 \cdot 10^{19}$ and $\log x_i < k_i \log m_1 + l_i \log m_2 < 9.726 \cdot 10^{11}$, that is, $x_i < \exp(9.726 \cdot 10^{11})$. This shows the Theorem in Case I.

Case II. We may assume that $x_1^2 + 1 < m_2$. We must have $l_1 = 0$ and $x_1^2 + 1 = 2m_1^{k_1} < m_2$.

Combining diophantine results mentioned in the proof of Lemma 4.1 allow us to assume that $k_1 = 1$ or 2 . Since (5) is nondegenerate, $l_i \neq 0$ for another index $i > 1$. Thus, $l_2, l_3 > 0$ and $x_2^2 + 1, x_3^2 + 1 > m_2$.

Now we clearly have

$$(28) \quad y_1 \arctan \frac{1}{x_1} \pm l_3 k_1 \arctan \frac{1}{x_2} \pm l_2 k_1 \arctan \frac{1}{x_3} = \frac{r\pi}{4}.$$

Let

$$(29) \quad \Lambda_1 = 2y_1 \log \frac{x_1 + \sqrt{-1}}{x_1 - \sqrt{-1}} - r\pi\sqrt{-1}.$$

Then, observing that $|r\pi/4 - y_1 \arctan(1/x_1)| < (1 + 10^{-8})k_1(l_2 + l_3)/m_2^{1/2}$ with $|y_1| \leq k_2 l_3 + l_3 k_2 < 2KL$, we have

$$(30) \quad |\Lambda_1| < \frac{4(1 + 10^{-8})k_1 L}{m_2^{1/2}},$$

while Théorème 3 of [3] gives that

$$(31) \quad -\log |\Lambda_1| < 8.87aH_1^2,$$

where $a = \max\{20, 10.98\widehat{\log} m_1 + (\log m_1)/2\}$ and $H_1 = \max\{17, 2.38 + \log((r/2a) + (2y_1/68.9))\}$.

We observe that $10.98\widehat{\log} N > 20$ for any N and therefore $a = 10.98\widehat{\log} m_1 + \frac{\log m_1}{2}$. Moreover, we have

$$(32) \quad \left| \frac{r\pi}{4} \right| \leq \frac{2KL}{x_1} + \frac{2u_1 L}{x_2} < \frac{2.3L}{m_1^{1/2}}.$$

and $|r| < 3KL/m_1^{1/2}$.

If $KL \geq 4 \cdot 10^7$, then, from (37), we obtain $\log m_2 < 8.87(10.98\gamma(m_1) + 0.51) \log m_1 \log^2(KL)$. If $m_2 \leq e^{187}$, then $m_1 < e^{187} < 4.14 \cdot 10^{81}$ and the Theorem immediately follows. If $m_2 > e^{187}$, then Lemma 4.1 yields that $KL < \log^{8.88163} m_2$. Observing that $8.87 \cdot 8.88163^2(10.98\gamma(m_1) + 0.51) \log m_1 > 14822.4$, we obtain

$$(33) \quad \log m_2 < 881.32(10.98\gamma(m_1) + 0.51) \log m_1 \log \log m_1.$$

Recalling (34), we have

$$(34) \quad \frac{2KL}{\sqrt{m_1 - 1}} > \frac{\pi}{4} - \frac{2k_1 L}{\sqrt{m_2 - 1}} > \frac{\pi(1 - 10^{-8})}{4}.$$

Combining this with (39) and Lemma 4.1, we have $m_1 < 4.14 \cdot 10^{81}$, $m_2 < \exp(9964497.86) < 2.7261 \times 10^{4327526}$, $\log x_i = k_i \log m_1 + l_i \log m_2 < 3.18 \cdot 10^{20}$, that is, $x_i < \exp(3.18 \cdot 10^{20})$, and $y_i \leq 2KL < 2.7 \cdot 10^{31}$. This completes the proof of the Theorem.

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